A method for determining and balancing the mass properties of the Free Flying Units

Ernest Company Vallet

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Nomenclature

F_i = force in each supporting point
W = weight
$\vec{r_i}$ = positions of the supporting points with respect to the geometric center of the setup
$\vec{R}$ = position of the Center of Gravity (CoG) with respect to the geometric center of the setup
T_i = tension in each cable
θ_i = vertical deflection angle of each cable
R_i = distance from each supporting point to the CoG
ϕ = horizontal angle of oscillation of the platform measured form its resting position
α_i = angle between two supporting points and the CoG

1 Introduction

A perfect balancing of both units (Rx and Tx) is needed in order for the experiment to be successful. In this case, a well-balanced unit has its Center of Gravity (CoG) placed on its geometrical center, its principal inertia axes are aligned with its geometrical axes, and the inertia distribution is symmetrical.

To achieve this balancing, the CoG and the tensor of inertia must first be measured, and then a set of balancing weights must be placed in order to bring the unit to the wanted configuration. A physical setup must be devised in order to measure the moments of inertia and to calculate the position of the CoG with as much accuracy as possible.

2 Measurements and Physical Setup

The setup devised for this task will serve both to calculate the CoG and to measure the Moments of Inertia. It will consist of a trifilar suspension pendulum, where the units will be placed in a plate hanging from a frame by means of three cables ($0.41mm$ thick Berkeley Fused Original fishing line, with a breaking weight of 40 kg) that are attached to said frame. An additional central support will hold most of the units' weight, greatly helping with the correct positioning of the units in the setup.

A sturdy, rigid frame is one of the main components of the physical setup for balancing, because it helps to hold all the rest of the components while not interacting with them in any way. The selected design consists of a triangular frame, composed of an upper triangle parallel to the floor supported by three triangular legs.

To make sure that the oscillations of the platform don't excite the torsional modes of the frame, a modal analysis was performed in the whole frame to characterize the natural frequencies corresponding to said torsional modes. Figure 1 shows the results; as it can be seen, the frequency corresponding to the torsional mode of the frame has a value of about 294.8 Hz. Since the oscillation frequency is in the range of 10 Hz, we can say that the platform's oscillation will not excite the frame.

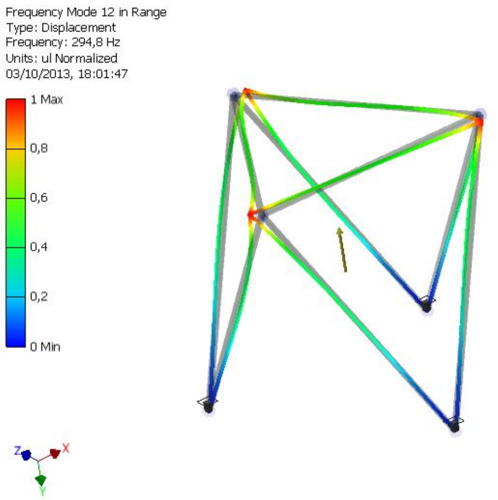

Figure 1: First torsional mode of the frame

A picture of the final setup is shown below, including the frame, the triangular platform, and several test masses used in the calibration procedures.

Figure 2: Picture of the completed measurement setup

2.1 Measuring the Center of Gravity

Once the unit is placed on the platform, the central support will come into play and replace one of the cables as the third support (resulting in the third cable becoming slack). Since the central support will be very close to the CoG of the unit, it will carry most of the weight by itself, thus leaving only a small amount of tension to the two remaining cables.

A diagram of the platform supporting the unit with its forces and arms is included below.

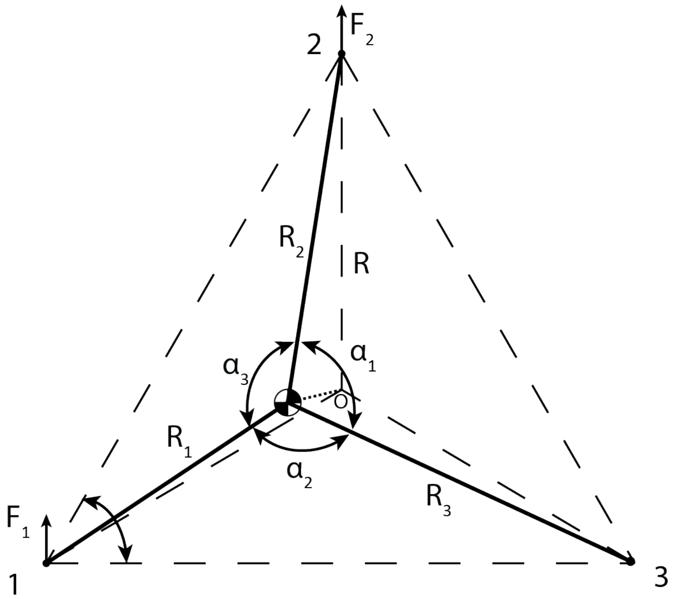

Figure 3: Force diagram of the platform

A precision scale will be used to make two measurements of the differential forces on the cables, one each time. Once these measurements are done, the position of the CoG can be determined using a static balance:

$$\begin{aligned} F_1 + F_2 + F_S &= W \\ \sum_i (\vec{r_i} - \vec{R}) \times \vec{F_i} &= 0 \end{aligned} \tag{2.1.1}$$

which is to say

$$\begin{aligned} F_1 + F_2 + F_S &= W \\ \sum_i \vec{r_i} \times \vec{F_i} = \vec{R} \times \sum_{i=1}^{3} \vec{F_i} &= \vec{R} \times \vec{W} \end{aligned} \tag{2.1.2}$$

Knowing the values of the positions $\vec{r_i}$ and the weight of the unit $\vec{W}$, and measuring the forces acting on the two supporting wires $\vec{F_1}$ and $\vec{F_2}$, the value of the CoG position $\vec{R}$ is fully determined in the XY plane. We do not need to know the value of the supporting force $\vec{F_S}$, as we are placing the reference frame in the geometric center of the platform where this force is applied, and therefore its arm is 0.

If we perform a simulation using a total weight of 4 kg, and we assume that the CoG is 3 mm away from the geometric center of the setup, we obtain that the tensions in the cables are in the order of 0.5% of the total weight, and thus the central support will take about 99% if the total force. The values of the tensions in the cables are around $0.1N$, which gives us the working range of the dynamometers/scales needed for the measurements.

2.2 Measuring the Tensor of Inertia

Once the CoG is known, the same setup can be used to determine the components of the tensor of inertia. This is done by letting the unit oscillate about a vertical axis and measuring the period of oscillation.

The three cables together have a total of 6 degrees of freedom. The platform they are supporting introduces a constraint that limits three of these degrees of freedom, which can be viewed as three fixed lengths. Therefore, these three degrees of freedom mean three possible eigenmodes for the oscillating system: one of them will be the torsional mode, and the other two are swaying motions in lateral directions.

When the platform is rotated about the vertical axis, there is a sideways resultant force driving it to the original configuration acting on each cable, with a value of

$$f_i = -T_i \sin\theta_i \tag{2.2.1}$$

with θ_i being the vertical angle that each particular cable has deflected and T_i the tension in each cable. Assuming small angles, the restoring torque that this resultant force produces in the disk is

$$\tau_i = -T_i R_i \sin\theta_i \approx -T_i R_i \theta_i \tag{2.2.2}$$

where R_i is the distance from each cable to the Center of Gravity.

Now, if we call ϕ the angle that the platform has rotated about its vertical axis, we know that $\sum_i \tau_i = I\ddot{\phi}$.

$$\sum_{i=1}^{3} \tau_i = -\sum_{i=1}^{3} T_i R_i \theta_i = I\ddot{\phi} \tag{2.2.3}$$

Since the three cables are not holding the same tension, nor the distances from each cable to the CoG are the same, we get that in the general case:

$$-(T_1R_1\theta_1 + T_2R_2\theta_2 + T_3R_3\theta_3) = I\ddot{\phi} \tag{2.2.4}$$

This can be rearranged in a more convenient way. If we denote α_i as the angle between two cables and the CoG , and we take into account that the torque about the line connecting each suspension point with the CoG should be zero, we can come up with a system of equations like the following:

$$\begin{aligned} T_1 + T_2 + T_3 &= W \\ T_1R_1\sin\alpha_3 - T_3R_3\sin\alpha_1 &= 0 \\ T_2R_2\sin\alpha_1 - T_1R_1\sin\alpha_2 &= 0 \end{aligned} \tag{2.2.5}$$

Solving for the tensions T_i, and plugging the result back into the original equation, we get:

$$-\frac{R_1R_2R_3W(\theta_1\sin\alpha_1 + \theta_2\sin\alpha_2 + \theta_3\sin\alpha_3)}{R_2R_3\sin\alpha_1 + R_1R_3\sin\alpha_2 + R_1R_2\sin\alpha_3} = I\ddot{\phi} \tag{2.2.6}$$

Since, for small angles, $R_i\phi = L\theta_i$, where L is the vertical length of the cables, we get:

$$I\ddot{\phi} + \frac{W}{L} f(R_i, \alpha_i)\phi = 0 \tag{2.2.7}$$

where

$$f(R_i, \alpha_i) = \frac{R_1R_2R_3(R_1\sin\alpha_1 + R_2\sin\alpha_2 + R_3\sin\alpha_3)}{R_2R_3\sin\alpha_1 + R_1R_3\sin\alpha_2 + R_1R_2\sin\alpha_3} \tag{2.2.8}$$

This is an equation for a harmonic oscillator, with an angular frequency of

$$\omega^2 = \frac{W}{IL} f(R_i, \alpha_i) = \frac{W}{IL}\frac{R_1R_2R_3(R_1\sin\alpha_1 + R_2\sin\alpha_2 + R_3\sin\alpha_3)}{R_2R_3\sin\alpha_1 + R_1R_3\sin\alpha_2 + R_1R_2\sin\alpha_3} \tag{2.2.9}$$

and a period of

$$T = \frac{2\pi}{\omega} \tag{2.2.10}$$

If we assume for simplicity that the distance to the CoG is the same for the three support points (i.e. the unit is perfectly centered on the platform), then the equation simplifies a lot, in fact we get

$$f(R_i, \alpha_i) = R^2 \tag{2.2.11}$$

and therefore, the equation 2.2.7 simplifies to:

$$I\ddot{\phi} + \frac{WR^2}{L}\phi = 0 \tag{2.2.12}$$

The angular frequency corresponding to this oscillator is

$$\omega^2 = \frac{WR^2}{IL} \tag{2.2.13}$$

and the period:

$$T = 2\pi\sqrt{\frac{IL}{WR^2}} \tag{2.2.14}$$

directly related to the inertia about the rotation axis. This allows us to calculate the inertia about the vertical rotation axis by measuring the period of oscillation of the unit. Putting in some numbers, assuming that both R and L have a value of 1 meter, and using a weight of the order of 1 kg and a moment of inertia of the order of 10^{-1} kgm^2 we can estimate the period of the oscillations to be of the order of $10^{-1}s$ to $1s$.

With this information, the three moments of inertia associated with the 3 geometrical axes of the unit (I_{XX}, I_{YY} and I_{ZZ}) can be measured. But a total of 6 different measurements are needed to get the 6 components of the Tensor of Inertia, as will be shown immediately.

In order to measure the products of inertia, the unit has to be inclined to a certain angle (45^o is the easiest); if we call A the axis about which the unit is now rotating, the general expression for the inertia about said axis A is:

$$I_A = I_{XX}\cos^2\alpha + I_{YY}\sin^2\beta + I_{ZZ}\sin^2\gamma - 2I_{XY}\cos\alpha\cos\beta - 2I_{YZ}\cos\beta\cos\gamma - 2I_{XZ}\cos\alpha\cos\gamma \tag{2.2.15}$$

where $\cos\alpha, \cos\beta$ and $\cos\gamma$ are the direction cosines that define the direction of the A axis. If this axis A is located for example at 45^o from the X and Y axes, in a horizontal plane, then the expression reduces to:

$$I_{XY} = \frac{I_{XX} + I_{YY}}{2} - I_A \tag{2.2.16}$$

This way, the 6 components of the tensor of inertia can be measured by calculating the products of inertia from these measurements at 45^o; we can then proceed to calculate the principal inertias as the eigenvalues of the matrix of inertia, and the principal axes as the eigenvectors associated with these eigenvalues.

Note that when we are measuring these moments of inertia, the values we get are the inertia of the unit plus the platform around their shared Center of Gravity; therefore, to get the correct value of the moment of inertia for the unit, we need to apply the parallel axis theorem each time we measure the moment of inertia about some axis, solving for I_{unit} :

$$I_{measured} = I_{unit} + I_{platform} + m_{platform}d^2_{CG,platform} + m_{unit}d^2_{CG,unit} \tag{2.2.17}$$

In this equation, $d_{CG,platform}$ and $d_{CG,unit}$ represent the distances from the Centers of Mass of both the platform and the unit with respect to the location of the common CoG.

As we said before, there are two additional degrees of freedom in the oscillating system that result in swaying about lateral directions. If we treat this swaying as a simple pendulum motion, we get:

$$mL^2\ddot{\varphi}(t) + mgL\varphi(t) = 0 \tag{2.2.18}$$

Solving the differential equation, with an initial condition $\varphi(0) = \varphi_0$, we get

$$\varphi(t) = \varphi_0 \cos\left(\frac{\sqrt{mgL}}{L\sqrt{m}}t\right) = \varphi_0 \cos\left(\sqrt{\frac{g}{L}}t\right) \tag{2.2.19}$$

which has a period of

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{2.2.20}$$

With a cable length of about 1 m, we get that the period of oscillation will have values around two seconds. Since this is an order of magnitude greater than the period of oscillation of the platform, we can safely say that the swaying motion will not interfere in the measurements of the units' inertia.

2.3 Working Procedure and Error Analysis

Once the physical setup is built, the measurements can begin. Error analysis plays a big part in the measurements, since we want to maximize the precision of the values we get. The first step is always to measure the Center of Gravity; before actually measuring the FFUs, a set of test pieces with a well-defined CoG will be used to check the accuracy of the measurements.

The equations that will be used have already been described in section 2.1, as well as the procedure: the platform is held by a central support and 2 cables, and measuring the forces in these cables (by means of a precision scale), the CoG is calculated.

Recall equation (2.2.5); the variables to be measured here are W, the two forces at the cables F_1 and F_2, and the positions of the cables with respect to the geometrical center of the platform $\vec{r}_i$.

It is better to keep the error definition in absolute terms, to keep the accuracy of the CoG placement in absolute units with physical meaning.

If we break equation 2.2.5 into components, the analysis goes as follows (it is the same for both X and Y components):

$$R_x = \frac{1}{W}(F_1 r_{1,x} + F_2 r_{2,x}) \tag{2.3.1}$$

and therefore, using propagation of uncertainty, we can see that

$$\Delta R = \left|\frac{\partial R}{\partial W}\right| \Delta W + \left|\frac{\partial R}{\partial F}\right| \Delta F + \left|\frac{\partial R}{\partial r}\right| \Delta r \tag{2.3.2}$$

Now, we can proceed to look at the precision of the different measurements. The weight of the unit will be determined using a weighing scale, which will at least have an accuracy of one gram. The determination of the positions of the cables (i.e. supporting points for the platform) can be easily done with a precision of one millimeter.

However, the accuracy of the precision scale will be a critical factor, since the range of the tensions in the cables goes around $0.1N$. Using a high-precision jeweler's scale, we can get down to precisions of up to 10^{-5} g.

The precision scale will be the adopted solution because of availability and accuracy reasons.

Table 1: Errors in the measurements of the CoG variables

Variable	$M(kg)$	$F_1(N)$	$r(m)$
Precision	0.001	10^{-4}	0.001
Relative Error (%)	0.02	0.1	0.1

The errors in the measurement of r can be further eliminated or reduced on a great scale by inserting them into the calibration process, i.e. by using an object with a well-known CoG and determining values for r to check the accuracy of the estimations. Therefore, the total absolute error in the measurements of the CoG will be in the order of 10^{-2} mm .

Once the CoG is known together with its uncertainty, we can proceed to the measurements of the moments of inertia. In this case, it is essential to run a calibration test before performing the actual measurements.

It is important to note that there are two different errors in these measurements: on the one hand, we have systematic errors, for example, in the values of fixed lengths that don't change throughout the process, that will be taken into account in the calibration procedure described in this section. On the other hand, there are random errors due for example to the measurement of the weight W and the period T that have to be dealt with.

Different test pieces will be used to perform a calibration of the test unit. One of them will be a hollow cylindrical steel piece with a weight of about 3.5 kg, since it is a simple, symmetrical object whose mass properties we can easily measure or calculate. The inertia for a hollow cylinder with external radius R, internal radius r and mass M about its central axis is determined by:

$$I_C = \frac{M}{2}(R^2 + r^2) \tag{2.3.3}$$

Following the error analysis for the CoG, we can proceed the same way here to determine the error for the calculation of the inertia given the radii and the mass of the cylinder. For a mass of 3.44 kg, and radii of 6.29 and 4.92 cm, and assuming an uncertainty in the measurements of 1 g for the mass and 0.1 mm for the distances, the uncertainty in the inertia is approximately $\Delta I_C = 4 \times 10^{-5}$ kg m^2, or a relative error of about 0.5%.

Once the error in the test piece is determined, we proceed to put it on the platform and let it oscillate around its vertical axis, letting it go freely, and measuring the period of oscillation. We also do the same while taking out the test piece, i.e. with an empty platform. Then we have two measurements of the period as follows:

$$\begin{aligned} I_P &= k\, g M_P T_P^2 \\ I_P + I_C &= k\, g(M_P + M_C) T_C^2 \end{aligned} \tag{2.3.4}$$

where M_P and M_C are the masses of the platform and the cylinder, and the same with the inertias. Solving this equation we can determine the value of I_P, the inertia of the platform, that we will need very soon. There are, however, two ways of dealing with k: either use the value that we get form these equation, or calculate the actual value using previous equations such as:

$$k = \frac{f(R_i, \alpha_i)}{(2\pi)^2 L} \approx \frac{R^2}{(2\pi)^2 L} \tag{2.3.5}$$

An expected value for the constant k based on the former definition:

$$k^{mean} \approx \frac{R^2}{(2\pi)^2 L} = 0.0253 \tag{2.3.6}$$

with extreme values such that:

$$\begin{aligned} k^{max} &\approx \frac{(R + \Delta R)^2}{(2\pi)^2 (L - \Delta L)} &= 0.0254 \\ k^{min} &\approx \frac{(R - \Delta R)^2}{(2\pi)^2 (L + \Delta L)} &= 0.0252 \end{aligned} \tag{2.3.7}$$

therefore

$$\Delta k = \frac{k^{max} - k^{min}}{2} = 0.00005 \tag{2.3.8}$$

Note that in this first estimation, all the masses (unit, platform) are assumed to have a common CoG in the geometrical center of the unit. This gives us an estimation for the value of k, but more realistic values will be obtained in the calibration.

Now we need to determine the accuracy of the inertia of the platform I_P. For this we follow again the same procedure as before, knowing that the value of this inertia is given by:

$$I_P = \frac{I_C M_P T_P^2}{(M_P + M_C) T_C^2 - M_P T_P^2} \tag{2.3.9}$$

As before, we include a table that lists the estimated precisions for the measurements and their relative error:

Table 2: Errors in the measurements of the Tensor of Inertia variables

Variable	M (kg)	T (s)	R (m)	L(m)
Precision	0.001	0.0001	0.001	0.001
Relative Error (%)	0.02	0.1	0.1	0.1

Now we can determine the uncertainty in I_P taking into account the uncertainty given by ΔI_C. Following the same procedure, we can arrive at an estimation for the value of ΔI_P that depends on the measured periods and mass of the platform. An approximated value for this uncertainty is $\Delta I_P \approx 3\times10^{-3}$ kgm^2, or a relative error of about 1.5 %.

Given the values from the calibration, we can now measure the actual moments of inertia of the FFUs. following the same procedure as with the cylinder test piece, we measure the period of oscillation of the system T_U and use the following equation:

$$\tilde{I}_P + \tilde{I}_U = \tilde{k}(\tilde{W}_P + \tilde{W}_U)\tilde{T}_U^2 \qquad (2.3.10)$$

where $\tilde{I}_P = I_P + \Delta I_P$ and so on. From this equation we can get the value of I_U, the inertia of the FFU assuming that the common Center of Gravity is in the geometric center (a more general case will be studied later).

Finally, we can ge the uncertainty in the inertia of the unit in the same way we operated with k:

$$I_U^{mean} = k(W_P + W_U)T_U^2 - I_P \qquad (2.3.11)$$

$$\begin{aligned} I_U^{max} &= (k + \Delta k)(W_P + W_U + 2\Delta W)(T_U + \Delta T)^2 - (I_P - \Delta I_P) \\ I_U^{min} &= (k - \Delta k)(W_P + W_U - 2\Delta W)(T_U - \Delta T)^2 - (I_P + \Delta I_P) \end{aligned} \qquad (2.3.12)$$

and then we obtain the value of ΔI_U by using:

$$\Delta I_U = \frac{I_U^{max} - I_U^{min}}{2} \qquad (2.3.13)$$

as with other cases above, it is difficult to estimate the values of some variables before actually testing them or having the real physical part built, therefore, we leave these estimations for the next section.

Once we know the values of the 3 inertias about the 3 geometric axes that pass through the CoG of the unit, the products of inertia are measured by tilting the unit 45^o, calculating the inertia with the same equations, and then using equation 2.2.15 to get the values of the 3 products of inertia. The error with the products of inertia has the same nature than the error in all the other inertias, and therefore it is of the order of ΔI_U^{CG} calculated before.

Now we have characterized the 6 components of the tensor of inertia of the FFU about its geometrical axes from its CoG. After all these considerations, we can proceed to calculate the principal inertias by determining the eigenvalues of the tensor of inertia, i.e. by solving for λ in the following equation:

$$det(I_{inertia} - \lambda I_{3x3}) = 0 \qquad (2.3.14)$$

where I_{3x3} is the identity matrix; then we can find the principal directions by determining the eigenvectors corresponding to these principal inertias, by solving this equation for v_i with each one of the three principal values:

$$\lambda_i v_i - I_{inertia} v_i = 0 \qquad (2.3.15)$$

where v_i is the eigenvector corresponding to the eigenvalue given by λ_i.

There is also an error associated with this transformation. Expanding equation 2.3.14, we get:

$$\begin{vmatrix} I_{XX} - \lambda & I_{XY} & I_{XZ} \\ I_{XY} & I_{YY} - \lambda & I_{YZ} \\ I_{XZ} & I_{YZ} & I_{ZZ} - \lambda \end{vmatrix} = 0 \qquad (2.3.16)$$

which is to say

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \qquad (2.3.17)$$

where

$$
\begin{aligned}
I_1 &= I_{XX} + I_{YY} + I_{ZZ} \\
I_2 &= I_{XX}I_{YY} + I_{XX}I_{ZZ} + I_{YY}I_{ZZ} - I_{XY}^2 - I_{XZ}^2 - I_{YZ}^2 \\
I_3 &= \begin{vmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{XY} & I_{YY} & I_{YZ} \\ I_{XZ} & I_{YZ} & I_{ZZ} \end{vmatrix}
\end{aligned} \tag{2.3.18}
$$

This is a cubic equation with a very complex analytical result; therefore, the error analysis can only be initiated once we have acquired all the rest of the numerical values for the tensor of inertia. The same happens with the principal directions.

2.3.1 Non-linear effects

Another possible source of error is the small angle assumption made back in equation 2.2.2, where we considered $\sin\theta \approx \theta$; the error with this assumption can be estimated running a numerical simulation comparing the two following cases:

$$
\begin{aligned}
I\ddot{\phi} + \frac{WR^2}{L}\phi = 0 \\
I\ddot{\phi} + \frac{WR^2}{L}\sin\phi = 0
\end{aligned} \tag{2.3.19}
$$

The analytic solution for the second equation involves elliptic integrals and is very complicated. A numerical solution has been calculated using Matlab, and it is shown in the figure below.

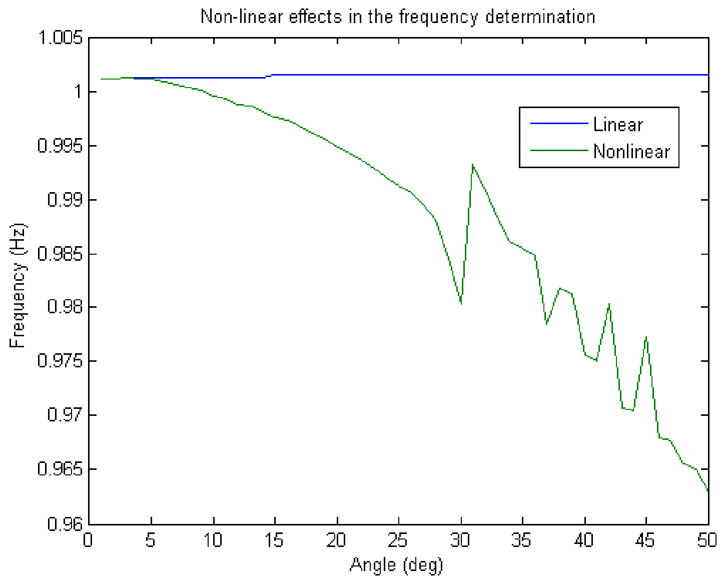

Figure 4: Nonlinear effects

In the simulation, different initial conditions (plotted in the x-axis) were used, to see the effect of the initial amplitude of the oscillation on the frequency. A frequency shift can be appreciated between the two solutions, with the non-linear one increasing its period and therefore decreasing its frequency.

This change is related to the amplitude used as initial condition. If this is small, then the difference between the two solutions is not perceptible. An initial angle less than 5° should not present a problem in terms of frequency shift.

3 Balancing weights

In order to achieve perfect balancing, a set of balancing weights have to be placed in the units to achieve a number of specifications, including:

- Minimize the total mass and number of the balancing weights
- Make sure that the CoG is placed as close as possible to the geometric center of the sections
- Ensure that I_{XX} equals I_{YY} as much as possible
- Get the principal inertia axes aligned with the geometrical axes (i.e. get the products of inertia as close to zero as possible)

To achieve these specifications, a set of constraining equations can be developed and forced onto the balancing weights. We can make a few assumptions beforehand; for example, we already have fixed locations for the balancing weights. This greatly reduces the number of parameters to be determined: we only need to know the mass and the angular position of the balancing weights.

To place the CoG in the geometric center of the units, we can define a set of coordinates in the center of the unit and calculate the new CoG after placing the balancing weights:

$$\vec{r}_{new} = \frac{\vec{r}_{old}M + \sum_{i=1}^{N} \vec{r}_i m_i}{M + \sum_{i=1}^{N} m_i} = \vec{0} \tag{3.0.20}$$

where r_i and m_i are the positions and the masses of the balancing weights.

Then, to make sure that the products of inertia get as close to zero as possible and to have I_{XX} as similar to I_{YY} as possible, we can develop another set of equations. If we call frame 0 the unit frame, and frame 1 the balancing weights frame, we first get:

$$I_i^0 = R_1^0 I_i^1 R_0^1 \tag{3.0.21}$$

where R describes the rotation matrix between the two frames. We have to use this equation because the balancing weights will have their inertia expressed initially in their respective body frames, unless spherical weights are used. Then, we have to use the parallel axes theorem in order to add the inertias of the unit and the balancing weights:

$$I_{total}^0 = I_{unit}^0 + I_i^0 + \sum_{i=1}^{N} m_i \begin{pmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & x_i^2 + z_i^2 & -y_i z_i \\ -x_i z_i & -y_i z_i & x_i^2 + y_i^2 \end{pmatrix} \tag{3.0.22}$$

where x_i, etc. describe the coordinates of the balancing weights CoG's with respect to the platform center.

All these constraints can be put as a function of the angular position of the balancing weights, so that the only unknowns are the mass and the angular distribution along the units. An algorithm to determine these parameters will be developed and tested using the CAD models as a reference.

4 Calibration results

Once the setup is ready, the calibrations to determine things such as the inertia of the platform can begin. First of all, the method used to determine the frequency of the oscillations is described.

With the platform performing a periodic oscillation, we put several tracking points (black dots) on its bottom side, and then we use a GoPro camera to record the oscillations and the movement of the tracking points. Afterwards, we use a tracking software (*Tracker*, an open-source program available online) to capture the movement of the tracking points and translate it into data that we can analyze. This data is then exported to Matlab; an example of the data can be seen below.

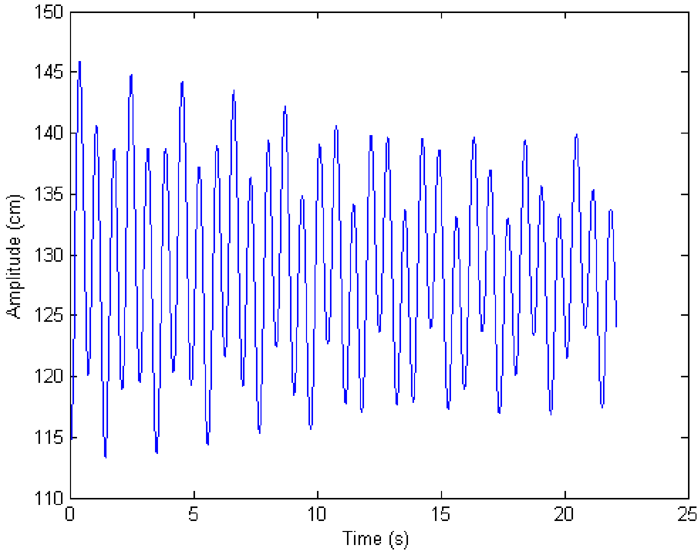

Figure 5: Data from tracking software

From this data, we want to extract the frequency of oscillation of the platform. If we look closely, we can notice that there are in fact to superposed oscillations in the data, one with a higher frequency than the other. This makes sense, for as it has been stated before, there are two principal modes of movement: an oscillation about the vertical axis, and a lateral swaying motion corresponding to the simple pendulum behavior.

In order to have an estimation of the two frequencies present in the oscillation, we do a Fourier transform of the data and observe the frequencies with the most power, as shown below.

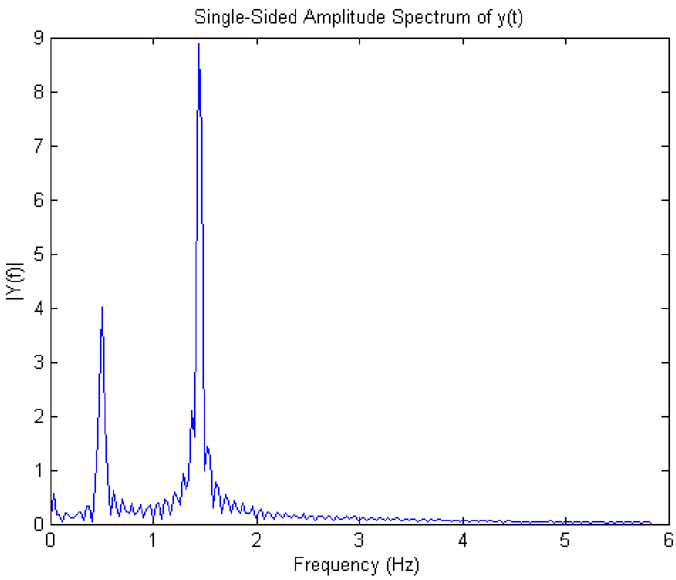

Figure 6: Fourier transform of the data

As we can see in the picture, there are clearly two main frequencies, with values around 0.5 and 1.4 Hz. The 0.5 Hz frequency is totally expected, since it corresponds to the pendulum motion and it can be calculated using

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \tag{4.0.23}$$

which, for a length of 1 m , yields a frequency of 0.498 Hz. The other frequency around 1.4 Hz is the one we are looking for; however, Fourier analysis is not precise enough for our purposes. Therefore, the next step is to try to fit the data into a known oscillation and extract the frequency from the fit.

We know that the data will fit in the sum of two damped sine waves with different amplitudes, frequencies, phases and damping coefficients. Thus, we will try to fit the data in this equation with 8 parameters:

$$x(t) = \sum_{i=1}^{2} e^{-\beta_i t} A_i \sin(2\pi f_i t + \phi_i) \tag{4.0.24}$$

Using the least squares fit package in Matlab, we do an 8-parameters fit using the obtained data and the frequencies from the Fourier transform as an initial guess; the results, which can be seen below, are very accurate in capturing all the oscillations in the data.

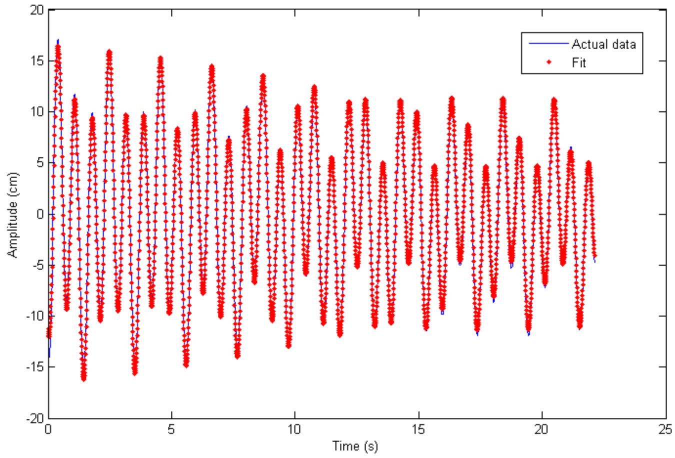

Figure 7: Least squares fitted curve

Now, we only have to look at the parameters of the least squares fit to know the frequencies; in this case, the main frequency is 1.443 Hz, and the secondary swaying frequency is 0.4986 Hz, as expected. The fit also gives us the standard error for the fit, which in the case of the frequency, has a value of $\sigma_f = 1.5 \times 10^{-5}\ s^{-1}$. Now that we have access to the oscillation frequency values, we can use equation 2.3.4 to calculate the values for k and I_P, which, using the previous results and knowing that the mass of the platform is 0.851 kg, gives us values of $k = 0.0239$ and $I_P = 0.2067\ kgm^2$. This value of k differs by 5% from the initial estimation of $k = (4\pi^2)^{-1}$; but repeated runs of the experiment show the same values for this constant.

Once the inertia of the platform is known, along with its Center of Gravity, we can do more measurements with different masses located off the center of the platform, in order to see the effect of an off-center CoG in the measurements. This will primarily affect the value of the constant k, since it is dependent on the position of the overall CoG.

The following picture (Figure 8) shows a contour plot for the values of k given the position of the system's common Center of Gravity, where we can see the value that we calculated just above in the position (0,0), corresponding to a perfectly balanced system, and then other values lower or higher depending on the location of said CoG.

This map can be used in the measurements and in fact has to be taken into account when getting the values for the object's inertia that we want to measure.

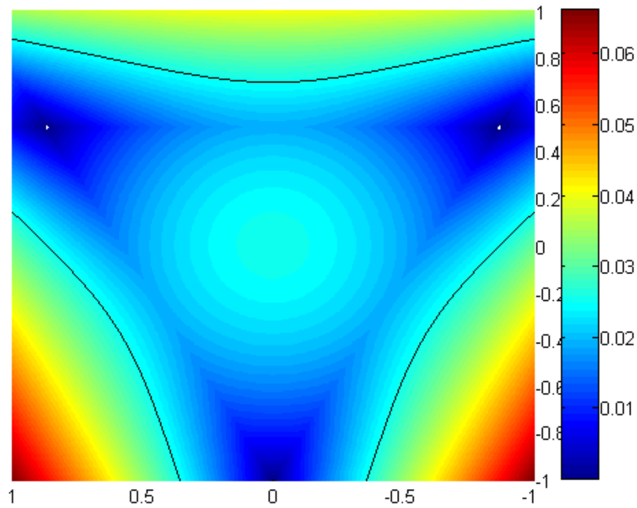

Figure 8: Value of k for different positions of the CoG

5 Measurement procedure

In this section we will clearly state the steps necessary to obtain measurements from the setup.

1. Clearly identify one point in the object to be measured, such as the geometrical center in case of a disk, and mark it.
2. Place the object in the platform, making sure that the geometrical centers of the platform and the object are vertically aligned.
3. Measure carefully the distances from the center to each supporting point of the platform, as well as the distances between the supporting points.
4. Raise the central spike so that the platform is now supported by only two cables and the center spike.
5. Use the precision scales to measure the forces in the cables by raising the scale slowly until it supports the corresponding weight of the platform.
6. Once the forces in the cables are known, the Center of Gravity can be calculated as described in section 2.1.
7. Then, proceed to twist the platform about 5° and release it so that it performs an oscillating movement.
8. Record the movement of several tracking points with the GoPro camera.
9. Use the Matlab data processing script to get the frequency measurements.
10. Using the calibration and the equations in section 2.2, calculate the moment of inertia taking into account the parallel axis theorem and the possibility that the CoG may be off-center.

6 Future work

With the setup prepared for the measurements, the next part of the process will be to use the knowledge of mass and inertia of the body to calculate a set of weights that is useful to provide balance to the unit. The testing of the balancing weights has fallen out of the scope of this project.